The BRST extension of gauge non-invariant Lagrangians

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Abstract. We show that, in gauge theory of principal connections, any gauge non-invariant Lagrangian can be completed to the BRST-invariant one. The BRST extension of the global Chern–Simons Lagrangian is present.

In perturbative quantum gauge theory, the BRST symmetry has been found as a symmetry of the gauge fixed Lagrangian [1]. The ghost-free summand L_1 of this Lagrangian is gauge non-invariant, but it is completed to the BRST-invariant Lagrangian $L = L_1 + L_2$ by means of the term L_2 depending on ghosts and anti-ghosts. We aim to show that any gauge non-invariant Lagrangian can be extend to the BRST-invariant one, though the anti-ghost sector of this BRST symmetry differs from that in [1].

In a general setting, let us consider a Lagrangian BRST model with a nilpotent odd BRST operator $\mathbf{s}$ of ghost number 1. Let L be a Lagrangian of zero ghost number which need not be BRST-invariant, i.e., $\mathbf{s}L \neq 0$. Let us complete the physical basis of this BRST model with an odd anti-ghost field σ of ghost number -1. Then we introduce the modified BRST operator

$$\mathbf{s}' = \frac{\partial}{\partial\sigma} + \mathbf{s} \tag{1}$$

which is also a nilpotent odd operator of ghost number 1. Let us consider the Lagrangian

$$L' = \mathbf{s}'(\sigma L) = L - \sigma\mathbf{s}L. \tag{2}$$

Since $\mathbf{s}$ is nilpotent, this Lagrangian is $\mathbf{s}'$-invariant, i.e., $\mathbf{s}'L' = 0$. Moreover, it is readily observed that any $\mathbf{s}'$-invariant Lagrangian takes the form (2). It follows that the cohomology of the BRST operator (1) is trivial.

Turn now to the gauge theory of principal connections on a principal bundle $P \to X$ with a structure Lie group G. Let VP and J^1P denote the vertical tangent bundle and the first order jet manifold of $P \to X$, respectively. Principal connections on $P \to X$ are represented by sections of the affine bundle

$$C = J^1P/G \to X, \tag{3}$$

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modelled over the vector bundle $T^*X\otimes V_GP$ [2]. Here, $V_GP=VP/G$ is the fibre bundle in Lie algebras $\mathfrak{g}$ of the group G. Given the basis $\{\varepsilon_r\}$ for $\mathfrak{g}$, we obtain the local fibre bases $\{e_r\}$ for V_GP. There is one-to-one correspondence between the sections $\xi=\xi^re_r$ of $V_GP\to X$ and the vector fields on P which are infinitesimal generators of one-parameter groups of vertical automorphisms (i.e., gauge transformations) of P. The connection bundle C (3) is coordinated by (x^μ,a^r_μ) such that, written relative to these coordinates, sections $A=A^r_\mu dx^\mu\otimes e_r$ of $C\to X$ are the familiar local connection one-forms, regarded as gauge potentials. The configuration space of these gauge potentials is the infinite order jet manifold $J^\infty C$ coordinated by $(x^\mu,a^r_\mu,a^r_{\Lambda\mu})$, $0<|\Lambda|$, where $\Lambda=(\lambda_1\cdots\lambda_k)$, $|\Lambda|=k$, denotes a symmetric multi-index. A k-order Lagrangian of gauge potentials is given by a horizontal density

$$L=\mathcal{L}(x^\mu,a^r_\mu,a^r_{\Lambda\mu})d^nx, \qquad 0<|\Lambda|\leq k, \qquad n=\dim X, \tag{4}$$

of jet order k on $J^\infty C$.

Any section $\xi=\xi^re_r$ of the Lie algebra bundle $V_GP\to X$ yields the vector field

$$u_\xi=u^r_\mu\frac{\partial}{\partial a^r_\mu}=(\partial_\mu\xi^r+c^r_{pq}a^p_\mu\xi^q)\frac{\partial}{\partial a^r_\mu} \tag{5}$$

on C where c^r_{pq} are the structure constants of the Lie algebra $\mathfrak{g}$. This vector field is the infinitesimal generator of a one-parameter group of gauge transformations of C. Its prolongation onto the configuration space $J^\infty C$ reads

$$J^\infty u_\xi=u_\xi+\sum_{0<|\Lambda|}d_\Lambda u^r_\mu\frac{\partial}{\partial a^r_{\Lambda\mu}}, \tag{6}$$

$$d_\Lambda=d_{\lambda_1}\cdots d_{\lambda_k}, \qquad d_\lambda=\partial_\lambda+a^r_{\lambda\mu}\frac{\partial}{\partial a^r_\mu}+a^r_{\lambda\lambda_1\mu}\frac{\partial}{\partial a^r_{\lambda_1\mu}}+a^r_{\lambda\lambda_1\lambda_2\mu}\frac{\partial}{\partial a^r_{\lambda_1\lambda_2\mu}}+\cdots. \tag{7}$$

A Lagrangian L (4) is called gauge-invariant iff its Lie derivative

$$\mathbf{L}_{J^\infty u_\xi}L=J^\infty u_\xi\rfloor d\mathcal{L}d^nx=(u_\xi+\sum_{0<|\Lambda|}d_\Lambda u^r_\mu\frac{\partial}{\partial a^r_{\Lambda\mu}})\mathcal{L}d^nx \tag{8}$$

along the vector field (6) vanishes for all infinitesimal gauge transformations ξ.

Let us extend gauge theory on a principal bundle P to a BRST model, similar to that in [3, 4]. Its physical basis consists of polynomials in fibre coordinates $a^r_{\Lambda\mu}$, $|0\leq\Lambda|$, on $J^\infty C$ and the odd elements C^r_Λ, $|0\leq\Lambda|$, of ghost number 1 which make up the local basis for the graded manifold determined by the infinite order jet bundle $J^\infty V_GP$ [5, 6]. The BRST operator in this model is defined as the Lie derivative

$$\mathbf{s}=\mathbf{L}_\vartheta \tag{9}$$

along the graded vector field

$$\vartheta = v^r_\mu \frac{\partial}{\partial a^r_\mu} + \sum_{0<|\Lambda|} d_\Lambda v^r_\mu \frac{\partial}{\partial a^r_{\Lambda\mu}} + v^r \frac{\partial}{\partial C^r} + \sum_{0<|\Lambda|} d_\Lambda v^r \frac{\partial}{\partial C^r_\Lambda}, \tag{10}$$

$$v^r_\mu = C^r_\mu + c^r_{pq} a^p_\mu C^q, \qquad v^r = -\frac{1}{2} c^r_{pq} C^p C^q,$$

where d_Λ is the generalization of the total derivative (7) such that

$$d_\lambda = \partial_\lambda + [a^r_{\lambda\mu} \frac{\partial}{\partial a^r_\mu} + a^r_{\lambda\lambda_1\mu} \frac{\partial}{\partial a^r_{\lambda_1\mu}} + \cdots] + [C^r_\lambda \frac{\partial}{\partial C^r} + C^r_{\lambda\lambda_1} \frac{\partial}{\partial C^r_{\lambda_1}} + \cdots]. \tag{11}$$

A direct computation shows that the operator $\mathbf{s}$ (9) acting on horizontal (local in the terminology of [3]) forms

$$\phi = \frac{1}{k!} \phi_{\alpha_1 \ldots \alpha_k} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$$

is nilpotent, i.e.,

$$\mathbf{L}_\vartheta \mathbf{L}_\vartheta \phi = \vartheta \rfloor d(\vartheta \rfloor d\phi) = [\sum_{0\leq|\Lambda|} \vartheta(d_\Lambda v^r_\mu) \frac{\partial}{\partial a^r_{\Lambda\mu}} + \sum_{0\leq|\Lambda|} \vartheta(d_\Lambda v^r) \frac{\partial}{\partial C^r_\Lambda}]\phi = 0.$$

Let L (4) be a (higher-order) Lagrangian of gauge theory. The BRST operator $\mathbf{s}$ (9) acts on L as follows:

$$\mathbf{s}L = (v^r_\mu \frac{\partial}{\partial a^r_\mu} + \sum_{0<|\Lambda|} d_\Lambda v^r_\mu \frac{\partial}{\partial a^r_{\Lambda\mu}}) \mathcal{L} d^n x, \qquad v^r_\mu = C^r_\mu + c^r_{pq} a^p_\mu C^q,$$

Comparing this expression with the expressions (5) and (8) shows that a Lagrangian L is gauge-invariant iff it is BRST-invariant. If L need not be gauge-invariant, one can follow the above mentioned procedure of its BRST extension. Let us introduce the anti-ghost field σ and the modified BRST operator $\mathbf{s}'$ (1). Then the Lagrangian

$$L' = L - \sigma \mathbf{s} L = L - \sigma(v^r_\mu \frac{\partial}{\partial a^r_\mu} + \sum_{0<|\Lambda|} d_\Lambda v^r_\mu \frac{\partial}{\partial a^r_{\Lambda\mu}}) \mathcal{L} d^n x \tag{12}$$

is $\mathbf{s}'$-invariant. If L is gauge-invariant, then $L' = L$. In particular, let L be a first order Lagrangian. Then its BRST extension (12) reads

$$L' = L - \sigma[(C^r_\mu + c^r_{pq} a^p_\mu C^q) \frac{\partial}{\partial a^r_\mu} + (C^r_{\lambda\mu} + c^r_{pq} a^p_{\lambda\mu} C^q + c^r_{pq} a^p_\mu C^q_\lambda) \frac{\partial}{\partial a^r_{\lambda\mu}}] \mathcal{L} d^n x. \tag{13}$$

For example, let us obtain the BRST-invariant extension of the global Chern–Simons Lagrangian. Let the structure group G of a principal bundle P be semi-simple, and let a^G be the Killing form on $\mathfrak{g}$. The connection bundle $C \to X$ (3) admits the canonical $V_G P$-valued 2-form

$$\mathfrak{F} = (da^r_\mu \wedge dx^\mu + \frac{1}{2} c^r_{pq} a^p_\lambda a^q_\mu dx^\lambda \wedge dx^\mu) \otimes e_r.$$

Given a section A of $C \to X$, the pull-back

$$F_A = A^*\mathfrak{F} = \frac{1}{2} F(A)^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r, \qquad F(A)^r_{\lambda\mu} = \partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq} A^p_\lambda A^q_\mu,$$

of $\mathfrak{F}$ onto X is the strength form of a gauge potential A. Let

$$P(\mathfrak{F}) = \frac{h}{2} a^G_{mn} \mathfrak{F}^m \wedge \mathfrak{F}^n$$

be the second Chern characteristic form up to a constant multiple. Given a section B of $C \to X$, the corresponding global Chern–Simons three-form $\mathfrak{S}_3(B)$ on C is defined by the transgression formula

$$P(\mathfrak{F}) - P(F_B) = d\mathfrak{S}_3(B)$$

[7]. Let us consider the gauge model on a three-dimensional base manifold X with the global Chern–Simons Lagrangian

$$\begin{aligned} L_{\rm CS} = h_0(\mathfrak{S}_3(B)) = [&\frac{1}{2} h a^G_{mn} \varepsilon^{\alpha\beta\gamma} a^m_\alpha (\mathcal{F}^n_{\beta\gamma} - \frac{1}{3} c^n_{pq} a^p_\beta a^q_\gamma) \\ &- \frac{1}{2} h a^G_{mn} \varepsilon^{\alpha\beta\gamma} B^m_\alpha (F(B)^n_{\beta\gamma} - \frac{1}{3} c^n_{pq} B^p_\beta B^q_\gamma) - d_\alpha (h a^G_{mn} \varepsilon^{\alpha\beta\gamma} a^m_\beta B^n_\gamma)] d^3x, \end{aligned}$$

where $h_0(da^r_\mu) = a^r_{\lambda\mu} dx^\lambda$ and

$$\mathcal{F} = h_0 \mathfrak{F} = \frac{1}{2} \mathcal{F}^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r, \qquad \mathcal{F}^r_{\lambda\mu} = a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu.$$

This Lagrangian is globally defined, but it is not gauge-invariant because of a background gauge potential B. Its BRST-invariant extension (13) reads

$$L'_{\rm CS} = L_{\rm CS} + h a^G_{mn} \sigma d_\alpha (\varepsilon^{\alpha\beta\gamma} (C^m_\beta a^n_\gamma + (C^m_\beta + c^m_{pq} a^p_\beta C^q) B^n_\gamma)) d^3x,$$

where d_α is the total derivative (11).

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